Indian Institute of Technology Kharagpur

Discrete Structures (CS21001) Endterm Examination, Autumn 2011

Date 22nd November, 2011 Full Marks: 90

Roll No	
Name	

Instructions:

- This booklet contains 20 pages.
- Your answers must be written in this question booklet as far as possible.
- You may use supplementary sheets only if required. If an answer is continued in the supplementary sheet, you must mention it in the relevant space of the question booklet.
- The supplementary sheets, if used, must contain your name and roll number, and tied with the question booklet.

Marks Obtained

1	
2	
3	
4	
5	
6	
7	
Total	

1. (a) A special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet two inhabitants: Ram and Mala. Ram tells you that Mala is a knave. Mala says, 'Neither Ram nor I are knaves.' So who is a knight and who is a knave? [5]

(b) Consider the following game: a pile of n stones is placed in front of two players. The players take turns removing 1, 2 or 3 stones from the pile. The player who removes the last stone loses the game. Prove that if n gives a remainder of 1 when divided by 4 then the second player has a winning strategy, i.e. can always win the game no matter how the first player plays. Also prove that in all other cases the first player has a winning strategy. [5]

2. (a) Show that among any 9 points inside a triangle of area 1 there are three points which form a triangle of area at most 1/4. [4]

(b) Prove with a combinatorial argument: [6]

i.

$$\sum_{k=0}^{n} C(n,k)^2 = C(2n,n)$$

ii.

$$\sum_{i=0}^{n} C(n,i) \cdot 2^i = 3^n$$

3. A function $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, m\}$ is called monotone non-decreasing if $1 \leq i \leq j \leq n \Rightarrow f(i) \leq f(j)$.

   (a) How many such functions are there? [4]

   (b) How many such functions are there that are surjective? [3]

   (c) How many such functions are there that are injective? [3]

4. (a) Let h_n denote the number of ways to tile a $2 \times n$ chessboard using 1×2 horizontal tiles, 2×1 vertical tiles, and 2×2 square tiles. Find a recurrence relation for h_n. [3]

   (b) Solve the recurrence relation $h_n = 3h_{n-1} - 2h_{n-2}$ with initial values $h_0 = 1$ and $h_1 = 2$. [3]

   (c) Let a_r denote the number of subsets of $\{1, 2, \ldots, r-1, r\}$ which do not contain two consecutive numbers. Find a_r. [4]

5. [5+5]

(a) Prove: If $a \equiv b \pmod{m}$ and $x \equiv y \pmod{m}$, then $ax \equiv by \pmod{m}$. [5]

(b) Prove that gcd is also a multiplicative function in a certain sense viz., If $gcd(b, c) = 1$ then $gcd(a, bc) = gcd(a, b)\ gcd(a, c)$ [5]

6. (a) Consider the set $2^{A \times A}$ of all binary relations on a set $A \neq \phi$. Prove that

   i. $2^{A \times A}$ is a monoid under relational composition. What is the identity element of the monoid? [5]

   ii. Given that one can define the inverse of a relation, explain why it is not a group. [5]

(b) i. Give a clear and complete statement of Lagranges Theorem for a subgroup H of a finite group G [5]

ii. Use Lagranges Theorem to prove that if $|G|$ is prime, then G must be cyclic. [5]

7. [5+5+5+5]

(a) At a party attended by n people, some pairs of party-goers shake hands and some do not. Prove that there must be two people with the same number of handshakes. [5]

(b) Let G be a simple (undirected) graph in which the minimum degree of any node is d. Prove that G contains a path of length d. [5]

(c) Prove that any simple (undirected) graph G with $d \geq 2$ has a cycle of length $\geq d+1$, where d is the minimum degree of any node in G. [5]

(d) In an old mansion which has only a single entrance, there is a ghost in every room which has an even number of doors. The ghosts are harmless but still scary. Prove that any visitor to the mansion can eventually find a room in which there are no ghosts. [5]

[Extra Page]