INDIAN INSTITUTE OF TECHNOLOGY KHARAGPUR
End-Spring Semester 2017-18

Date of Examination: 20 Apr, 2018 **Session:** AN **Duration:** 3 hrs **Full Marks:** 50
Subject No. : PH41008 **Subject:** MATHEMATICAL METHODS -II
Department/Center/School: PHYSICS
Specific charts, graph paper, log book etc. required: NONE
Special Instructions (if any): Any notes or books are NOT allowed

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Answer all questions.

1. State whether true or false, and provide a very brief justification for your answer.

   (a) The set of rational numbers, excluding zero, is a *group* under multiplication.

   (b) The set of rational numbers, excluding zero, is also a *field* under addition and multiplication.

   (c) The symmetry group of an isosceles triangle has 3 distinct elements.

   (d) The permutation group S_3 is a subgroup of S_6 but not a subgroup of S_{720}.

   (e) The quaternion group $\mathcal{Q}$ (of order 8) has a subgroup isomorphic to $\mathbb{Z}_3$.

   (f) In the group $V_4 = \mathbb{Z}_2 \otimes \mathbb{Z}_2$, different conjugacy classes corresponds to different cycle structures, since all elements in the same conjugacy class of a finite group, must have the same cycle structure.

   (g) Up to isomorphisms, there are exactly two distinct group structures for groups of order 89.

   (h) There exists an isomorphism between the groups $SU(2)$ and $SO(3)$.

   (i) Two irreducible representations can never have the same dimension.

   (j) The number of possible irreducible representations of a group of order 11 is 10.

   [10 Marks]

2. (a) Consider the following four functions

   $$f_1(x) = x,\ \ f_2(x) = -x,\ \ f_3(x) = \frac{1}{x}, f_4(x) = -\frac{1}{x}.$$

   Does this set of four functions form a group under the operation $f_1 \circ f_2 = f_1(f_2(x))$? If it forms a gourp, what is the order of each of these four elements?

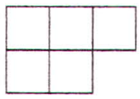
   (b) Write down the group composition table (Cayley table) of the group of order 6, with the presentation

   $$\langle a, b \mid a^3 = e = b^2, bab^{-1} = a^{-1} \rangle$$

   (c) Prove that, in any finite group with *even* order, there must be at least one element, other than identity, which is inverse of itself. [10 Marks]

3. (a) Find the various possible cycle structures of S_5 (group of permutation of 5 objects), and the total number of elements corresponding to every cycle structure.

(b) Which cycle of S_5 corresponds to the young diagram



(c) If two elements in S_5 is $g_1 = (12)(34)(5)$ and $g_2 = (32415)$, what is $g_1 \circ g_2$ under the usual group operation. Also, find the conjugate element of the element g_2 with respect to g_1.

[10 Marks]

4. (a) Obtain the left and the right cosets of S_3 with respect to its subgroup $\mathbb{Z}_2 = \{e, (12)\}$. From this result, is it possible to deduce whether S_3 is simple or not?

(b) Find the factor group $\mathbb{Z}_6/\mathbb{Z}_2$.

(c) Prove that a subgroup $\mathcal{H}$ of the finite group $\mathcal{G}$, which contains half of all elements in $\mathcal{G}$ (i.e. order of $\mathcal{G} = 2 \times$ order of $\mathcal{H}$), must be a normal (invariant) subgroup.

[10 Marks]

5. (a) Work out the character table of the group of symmetries of the equilateral triangle $\mathcal{D}_3$. At first, explicitly separate the elements of this group into conjugacy classes and then clearly provide the full justification behind every entry of the character table.

(b) Consider the action of the group $\mathcal{D}_3$ on the two dimensional subspace spanned by the functions $\psi_1 = \cos n\theta, \psi_2 = \sin n\theta$, n being a positive integer, and θ is the polar angle, with the origin chosen as the center of the equilateral triangle. Now consider the representation D, of the group $\mathcal{D}_3$, on this two dimensional space. In this representation, one of the rotations (R) and one of the reflections (ρ) are represented by the matrices

$$D(R) = \begin{pmatrix} \cos\frac{2}{3}n\pi & -\sin\frac{2}{3}n\pi \\ \sin\frac{2}{3}n\pi & \cos\frac{2}{3}n\pi \end{pmatrix}, \quad D(\rho) = \begin{pmatrix} -\cos n\pi & 0 \\ 0 & \cos n\pi \end{pmatrix}$$

From this information (and the result of part(a) above), write this representation D as a direct sum of irreducible representations of $\mathcal{D}_3$. Hence deduce, for what values of n is the representation D reducible or irreducible.

[10 Marks]