INDIAN INSTITUTE OF TECHNOLOGY, KHARAGPUR

End-Autumn Semester Examination-2018-19

Date of Examination:15.11.2018 Session: AN Duration: 3hrs Full Marks: 50
Subject No: MA20105 Subject: Linear Algebra
Department/Center/School: Mathematics

Instructions: Answer ALL questions. Write all the parts of the same question together.

1. Answer **ALL** parts.

(a) Consider the vector spaces $V = P_1(\mathbb{R})$ and $W = \mathbb{R}^2$ together with their standard bases β and γ respectively. Define $T : V \longrightarrow W$ by

$$T(p(x)) = (p(0) - 2p(1), p(0) + p'(0))$$

where $p'(x)$ is the derivative of $p(x)$. Let $T^* : W^* \longrightarrow V^*$ denotes the corresponding linear transformation between the dual spaces W^* and V^*.
(i) Find the matrix of T^* with respect to the dual bases β^* and γ^*.
(ii) For $f \in W^*$, defined by $f(a, b) = a - 2b$, compute $T^*(f)$. [3M]

(b) Let T be a linear operator on an inner product space V and suppose that $||T(x)|| = ||x||$ for all $x \in V$. Is T one to one? Justify your answer. [2M]

(c) Let $T : P_2(\mathbb{R}) \longrightarrow P_4(\mathbb{R})$ defined by $T(f)(x) = f(x^2)$. Determine if T is a linear transformation. If it is, find the matrix representation for T relative to the basis $\beta = \{1, x, x^2\}$ of $P_2(\mathbb{R})$ and basis $\gamma = \{1, x, x^2, x^3, x^4\}$ of $P_4(\mathbb{R})$.
[3M]

(d) Let W be the subspace of $\mathbb{C}^3$ spanned by $\{(1, 0, i), (1, 2, 1)\}$. Compute $W^{\perp}$.
[2M]

2. Answer **ALL** parts.

(a) Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix. If A is non-singular then show that

$$|M| = |A| \ |D - CA^{-1}B|.$$

[4M]

(b) Let $M = \begin{pmatrix} A & B \\ C & O \end{pmatrix}$ be a block matrix, where B and C are non-singular, and O is the zero matrix. Find M^{-1} in block form. [3M]

(c) Let $Z = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be a block matrix where A and D are square matrices. If $Z^{-1} = \begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ then show that

$$|S| = \frac{|A|}{|Z|} \qquad |P| = \frac{|D|}{|Z|}$$

[3M]

3. Answer **ALL** parts.

   (a) Let A be a square matrix and λ be an eigenvalue of A. Show that the algebraic multiplicity of λ is not smaller than its geometric multiplicity. [4M]

   (b) Prove that every eigenvalue of a square matrix A is a root of the minimal polynomial of A. [3M]

   (c) Prove that the chain generated by a generalized eigenvector of type m corresponding to an eigenvalue of a matrix is a linearly independent set. [3M]

4. Answer **ALL** parts.

   (a) State and prove the Cayley-Hamilton theorem. [4M]

   (b) Find $A^{593} - 2A^{15}$, for $A = \begin{pmatrix} -2 & 4 & 3 \\ 0 & 0 & 0 \\ -1 & 5 & 2 \end{pmatrix}$. [3M]

   (c) Let α be a scalar which is not an eigenvalue of a square matrix A. Let $B = (A - \alpha I)^{-1}$. Find all the eigenvalues and eigenvectors of B in terms of eigenvalues and eigenvectors of A. [3M]

5. Answer **ALL** parts.

   (a) For the matrix $A = \begin{pmatrix} 2 & 1 & -2 \\ 2 & 3 & -4 \\ 1 & 1 & -1 \end{pmatrix}$,

      (i) Find all the eigenvalues and dimension of their eigenspaces.

      (ii) Find a non-singular matrix P (if exists) such that $P^{-1}AP$ is a diagonal matrix. [4M]

   (b) Consider the matrix $A = \begin{pmatrix} 2 & 1 & 0 & -1 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. Then

      (i) Find a canonical basis for A. [3M]

      (ii) Find Jordan canonical form of A. [1M]

      (iii) Find the minimal polynomial of A. [2M]

****************THE END****************